TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 261, Number 1, September 1980

UNIFORM APPROXIMATION ON UNBOUNDED SETS BY HARMONIC FUNCTIONS WITH LOGARITHMIC SINGULARITIES[1]

BY

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ABSTRACT. This paper deals with the qualitative theory of uniform approximation by harmonic functions. The theorems of Brelot and Deny on Runge- and Walsh-type approximation on compact sets are extended to unbounded closed sets.

1. Introduction. Suppose h is harmonic in a deleted neighborhood of a finite point $a \in \mathbf{R}^2$. We say that h has a logarithmic singularity at a if there is a function u harmonic at a, and a constant α such that in a neighborhood of a, $h - u$ is of the form

$$\alpha \ln|z - a|.$$

Since this notion is conformally invariant, it also makes sense on a Riemann surface R. An *essentially harmonic function* on R is a function which is harmonic except possibly for logarithmic singularities. It follows from Lemma 4 below that an essentially harmonic function can be written as the difference of subharmonic functions.

Without loss of generality we may and shall assume that every Riemann surface R is connected. Following Scheinberg [12] we say that a subset is *bounded* in R if its closure in R is compact. A Riemann surface R' is said to be an *extension* of R if R is (conformally equivalent to) an open subset of R'. If furthermore $\bar{R} \neq R'$, R' is an *essential extension* of R. We shall say that a closed subset of R is *essentially of finite genus* if F has a covering by a family of pairwise disjoint open sets, each of finite genus. We may assume that such a cover is locally finite.

Throughout this paper, R denotes an open Riemann surface and R^* its one point compactification. Unless otherwise specified, all topological notions refer to R. Thus, if F is a subset of R, F^- denotes the R-closure of F, ∂F is the R-boundary, etc.

In this paper we establish the following results:

THEOREM 1. *Let F be closed and essentially of finite genus in an open Riemann surface R. Then, each function essentially harmonic on F is the uniform limit of functions essentially harmonic on R.*

Received by the editors May 8, 1979 and, in revised form, September 4, 1979. Presented at the conference in Classical Complex Analysis, Purdue University, 1980.

1980 *Mathematics Subject Classification*. Primary 30E10, 30F15, 31A05.

[1]Research supported in part by N.S.F. grants MC 576-82924 and MSC 77-00962, by the N.R.C. of Canada, and by the Ministère d'Education du Québec.

Let us call the approximation in Theorem 1 a *Runge-type approximation* by essentially harmonic functions. A Runge-type approximation by harmonic functions is defined analogously by replacing essentially harmonic functions by harmonic functions. Denote by $\hat{F}$ the union of F and all bounded components of $R \setminus F$.

THEOREM 2. *Let F be closed in an open Riemann surface R. The following conditions are necessary in order for a Runge-type approximation by harmonic functions to be possible on F.*

(I) *$R^* \setminus \hat{F}$ is locally connected.*

(II) *For each bounded open set V such that $\partial V \subset F$, either $V \subset F$ or $V \cap F = \varnothing$.*

(III) *For each compact set K in R, there is a compact set K' in R which contains every bounded component of $R \setminus (F \cup K)$ whose closure meets K.*

THEOREM 3. *Let F be closed in an open Riemann surface R. Consider the condition*:

(IV) *$R^* \setminus F$ is connected and locally connected.*

If F is essentially of finite genus in R, then (IV) *is sufficient for Runge-type harmonic approximation on F.*

Let G be an open set in R. We say that (G, R) is a *Runge pair* for approximation on closed sets if for each h harmonic on G, each subset F of G which is R-closed, and each $\varepsilon > 0$, there exists a u harmonic on R such that $|h - u| < \varepsilon$ on F.

THEOREM 4. *Let R be an open Riemann surface. In order for (G, R) to be a Runge pair for harmonic approximation on closed sets, it is necessary that $R^* \setminus G$ be connected. If G is essentially of finite genus, this condition is also sufficient.*

Let F be closed in an open Riemann surface R. We say that *Walsh-type approximation* by (essentially) harmonic functions is possible for the pair (F, R) if every function continuous on F and harmonic on F° can be approximated uniformly on F by functions (essentially) harmonic on R. Of course, the necessary conditions for Runge approximation are all the more necessary for Walsh approximation.

THEOREM 5. *Let F be closed and essentially of finite genus in an open Riemann surface R. Suppose $F = F^{\circ -}$ and ∂F is analytic. Then F is a set of Walsh approximation by essentially harmonic functions. If, moreover, $R^* \setminus F$ is connected, then F is a set of Walsh approximation by harmonic functions.*

We remark that conditions (I) to (IV) are not independent of each other. More will be said on this later. For the present we feel it is better to retain all of these conditions, despite the redundancy, in order to grasp more easily the geometric relationship between F and R.

In case F is compact, our results are implicit in the work of Brelot [2] and Deny [3], at least when R is a planar domain. Hence the thrust of our work is that we can now approximate on unbounded sets. Of course, from the compact theorems it follows that one can approximate uniformly *on compact subsets of* closed sets. But we are doing much more. For example, from Theorem 1 it follows that if R is a plane domain, F is R-closed, h is harmonic on F, and ε is positive, then there is a u essentially harmonic on R such that $|h - u| < \varepsilon$ everywhere on F. Hence, the approximation is uniform simultaneously on all of F, not just on compact subsets.

It should also be pointed out that we make no assumption about the behavior near "infinity" of the function to be approximated. It could oscillate wildly.

Such theorems are already known for holomorphic or meromorphic approximation and have borne rich fruits (see for example [3]). If R is simply connected the sufficiency in Theorem 3, for instance, is a trivial consequence of the holomorphic analogue obtained by passing to the holomorphic completion of a harmonic function. However, if R is not simply connected, we see no way of deducing the harmonic results from the holomorphic analogues.

The only paper we are aware of which deals with the subject of this paper is the work by Saginjan [11]. In that paper Saginjan works in $\mathbf{R}^n$, but restricts his investigation to closed subsets F with no interior. Thus, he is approximating continuous functions by harmonic functions.

We do not know whether our own results extend to $\mathbf{R}^n$, and to arbitrary open Riemann surfaces. However, the proofs would need to be modified in the case of $\mathbf{R}^n$ as we make use of conformal mappings. Moreover, we note that the holomorphic analogues fail on arbitrary open Riemann surfaces [5] and in $\mathbb{C}^n$, $n > 1$.

ADDED IN PROOF. We have recently extended some of our results to $\mathbf{R}^n$, $n > 2$.

2. Preliminaries. Let K be a compact set in an open Riemann surface R, and let f be in $C(\partial K)$, the class of continuous (real-valued) functons on ∂K. We may extend f continuously to a neighborhood of K. Let $\{\Omega_n\}$ be a sequence of neighborhoods of K nesting down on K and such that each Ω_n is regular for the Dirichlet problem. Denote by $H^n(f)$ the solution of the Dirichlet problem on Ω_n with boundary function $f|\partial\Omega_n$. Recall that a point $x \in \partial K$ is called *stable* for K if $H^n(f)(x) \to f(x)$ for each $f \in C(\partial K)$. It is well known that x is stable for K if and only if $R \setminus K$ is not thin at x [2]. Denote by $S(K)$ the points of ∂K which are stable for K.

LEMMA 1. *Let K be compact and $f \in C(K)$. Then*

$$H^n(f) \to f$$

uniformly on compact subsets of $S(K)$.

Since f can be uniformly approximated by a difference of continuous subharmonic functions [6, p. 196], we may assume that f itself is subharmonic as well as continuous. Thus, $H^n(f)$ decreases to f on $S(K)$. The lemma now follows from Dini's theorem [6, p. 35].

Let A be a subset of an open Riemann surface R and denote by $\hat{A}$ the *envelope* of A. That is, $\hat{A}$ is the union of A and all bounded components of $R \setminus A$. The set A is called *full* (*plein* in French) in R if it is equal to its envelope. If A is closed, compact, or open, then $\hat{A}$ is closed, compact, or open, respectively. If $A \subset B$, then $\hat{A} \subset \hat{B}$. For references to proofs and to the historical development of this notion, see [9].

LEMMA 2. *A is full iff $R^* \setminus A$ is connected.*

PROOF. Suppose first that A is full, and let V be a component of $R^*\setminus A$. It suffices to show that the ideal point ∞ is in V. Suppose $\infty \notin V$. Thus, V is a component of $R\setminus A$. $\overline{V}$ is not compact (since A is full), and so V is contained in no compact set. Hence, ∞ is in the R^*-closure of V. Since V is a component of $R^*\setminus A$, it follows that $\infty \in V$, which is the desired contradiction.

Conversely, suppose $R^*\setminus A$ is connected, and let V be a component of $R\setminus A$. It is enough to show that V is not bounded. Now V is open and closed in $R\setminus A$. Thus, V is open in $R^*\setminus A$. If V were bounded, then V would also be closed in $R^*\setminus A$. Since $R^*\setminus A$ is connected, V would have to be all of $R^*\setminus A$ which is not possible since $\infty \notin V$. This proves the lemma.

Note that the properties we have discussed for $\hat{A}$ hold for more general spaces R than we are considering.

Let F be full in an open Riemann surface R. We shall say that an exhaustion

$$R = \bigcup_{j=1}^{\infty} R_j$$

of R is *compatible* with F if $F \cup \overline{R_j}$ is full in R for each j.

LEMMA 3. *If $R^*\setminus F$ is connected and locally connected, then there is an exhaustion of R compatible with F.*

PROOF. Let $\{G_j\}$ be a nested exhaustion of R by bounded full open sets. Since F is full, there is a $G_{j(1)}$ such that

$$\left(\overline{G}_j \cup F\right)^{\wedge} \subset G_{j(1)} \cup F.$$

Similarly, there is a $G_{j(2)}$ such that

$$\left(\overline{G}_{j(1)} \cup F\right)^{\wedge} \subset G_{j(2)} \cup F.$$

We choose inductively a subsequence $G_{j(k)}$, $j(k) < j(k+1)$, $k = 1, 2, \ldots$, such that

$$\left(\overline{G}_{j(k)} \cup F\right)^{\wedge} \subset G_{j(k+1)} \cup F.$$

Now set

$$R_k = \left(\overline{G}_{j(k)} \cup F\right)^{\wedge\circ} \cap G_{j(k+1)},$$

for $k = 1, 2, \ldots$. Then $\{R_k\}$ is an exhaustion by full bounded open sets and $F \cup \overline{R}_k$ is full. Hence, the exhaustion $\{R_k\}$ is compatible with F.

The following Cousin-type lemma shows that in order to approximate essentially harmonic functions, it is enough to know how to approximate harmonic functions.

LEMMA 4 (PFLUGER [8, P. 194]). *Let $\{R_j\}$ be an open cover of an open Riemann surface R. If p_j are essentially harmonic on R_j and $p_j - p_k$ are harmonic on $R_j \cap R_k$ for all j and k, one can find p essentially harmonic on R so that $p - p_j$ is harmonic on R_j for every j.*

For the next lemma, we shall say that a function h is *essentially a C^2-function* on R if

$$h = p + \varphi, \tag{1}$$

where p is essentially harmonic and φ is a C^2-function.

LEMMA 5 (GREEN FORMULA). *Suppose D is a bounded domain with C^1-boundary in an open Riemann surface R. If h is essentially a C^2-function on $\overline{D}$, then we can write*

$$h(z) = r(z) - \frac{1}{2\pi} \iint_D g(\zeta, z)\Delta\varphi(\zeta)\, d\xi\, d\eta \tag{2}$$

where r is essentially harmonic on D, g is the Green function for D, φ is given by (1), *$\zeta = \xi + i\eta$, and $\Delta\varphi(\zeta)d\xi d\eta$ is invariantly defined.*

PROOF. (2) is an immediate consequence of (1) and the well-known Green representation formula for C^2-functions:

$$\varphi(z) = u(z) - \frac{1}{2\pi} \iint_D g(\zeta, z)\Delta\varphi(\zeta)\, d\xi\, d\eta$$

where u is harmonic.

The compact version of our Runge-type Theorem 1 was proved for plane domains by Brelot [**2**] and Deny [**3**]. We need the analogous theorem on open Riemann surfaces.

LEMMA 6 (RUNGE-TYPE). *Let K be a compact subset of an open Riemann surface R. Then each function h essentially harmonic on K is the uniform limit of functions essentially harmonic on R. If K is full in R and h is harmonic on K, we may take the approximating function to be harmonic on R.*

PROOF. The case where K is full is due to Pfluger [**8**, p. 192] (see also [**7**, p. 347]). Let h be essentially harmonic on K and $\varepsilon > 0$. Then by the Cousin-type Lemma 4, there is a function h_1 essentially harmonic on R with $h_1 - h$ harmonic on K. Let G be a neighborhood of K which is full in R and bounded by analytic Jordan curves. An argument of Pfluger [**8**, p. 194] shows that $h_1 - h$ can be approximated within $\varepsilon/2$ on K by a function v_0 essentially harmonic on G. Now by Pfluger's Runge-type theorem [**8**, p. 192], v_0 can be approximated within $\varepsilon/2$ on K by a function v_1 harmonic on R. Set $v = h_1 - v_1$. Then v is essentially harmonic on R and approximates h within ε. This completes the proof.

3. Fusion lemma. Alice Roth [**10**] first proved a fusion lemma for rational approximation. In this section we shall develop a harmonic analogue.

LEMMA 7. *Suppose δ and x_0 are positive. Then, there is a positive constant $a = a(\delta, x_0)$ such that if h is harmonic and $|h| < \varepsilon$ for $|x_0 - x| < \delta$, then there is a C^2-function v such that*

$$v = 0, \quad \text{for } x \leqslant 0,$$
$$|v| < a\varepsilon, \quad \text{for } 0 \leqslant x \leqslant x_0,$$
$$v = h, \quad \text{for } x_0 \leqslant x < x_0 + \delta,$$
$$|\Delta v| < a\varepsilon, \quad \text{whenever } v \text{ is defined.}$$

PROOF. We shall construct v of the form

$$v(x, y) = \sum_{i=0}^{5} a_i(y)x^i, \qquad 0 \leqslant x \leqslant x_0.$$

The boundary conditions are

$$v(0,y) = v_x(0,y) = v_{xx}(0,y) = 0, \tag{1}$$

$$\left.\begin{aligned} v(x_0, y) &= h(x_0, y), \\ v_x(x_0, y) &= h_x(x_0, y), \\ v_{xx}(x_0, y) &= h_{xx}(x_0, y). \end{aligned}\right\} \tag{2}$$

From (1) we have $a_0 = a_1 = a_2 = 0$.
From (2) we have

$$x_0^3 a_3(y) + x_0^4 a_4(y) + x_0^5 a_5(y) = h(x_0, y),$$
$$3x_0^2 a_3(y) + 4x_0^3 a_4(y) + 5x_0^4 a_5(y) = h_x(x_0, y),$$
$$6x_0 a_3(y) + 12x_0^2 a_4(y) + 20x_0^3 a_5(y) = h_{xx}(x_0, y).$$

This system has a solution a_3, a_4, a_5. Each a_j is a linear combination of $h(x_0, y)$, $h_x(x_0, y)$, and $h_{xx}(x_0, y)$.

Fix a value y_0. In the disc $D(y_0)$ of center (x_0, y_0) and radius $\delta/2$, the functions h, h_x, and h_{xx} have integral representations obtained by differentiating the Poisson formula for h. Since h is bounded by ε in $|x_0 - x| < \delta$, there is a constant A independent of y_0 such that $h(x_0, y_0)$, $h_x(x_0, y_0)$, $h_{xx}(x_0, y_0)$ are bounded by $A\varepsilon$. Hence the a_j are also bounded by a constant times ε and the same is true for v on $0 \leqslant x \leqslant x_0$. Since the a_j are bounded by a constant times ε, it also follows that the same holds for Δv.

There only remains to verify that v is a C^2-function, and this need only be checked on the lines $x = 0$ and $x = x_0$. Along these lines the partial derivatives are continuous from the left and continuous from the right and therefore continuous.

LEMMA 8. *Let $x_0 > 0$ and suppose U is a neighborhood of the segment $x = x_0$, $0 \leqslant y \leqslant y_0$. Then there is a positive constant $a = a(U)$ such that if h is harmonic and bounded by ε on U, then there is a v as in the previous lemma, but now v is only defined on*

$$(x \leqslant 0) \cup (0 \leqslant x \leqslant x_0, 0 \leqslant y \leqslant y_0) \cup (U \cap (x > x_0)).$$

The proof is identical to that of the previous lemma.

LEMMA 9. *Suppose $\delta > 0$ and $\rho > 1$. Then there is a positive constant $a = a(\delta)$ such that if h is harmonic and $|h| < \varepsilon$ for $||z| - \rho| < \delta$, then there is a C^2-function v such that*

$$v = 0, \quad \textit{for } |z| \leqslant 1,$$
$$|v| < a\varepsilon, \quad \textit{for } 1 \leqslant |z| \leqslant \rho,$$
$$v = h, \quad \textit{for } \rho \leqslant |z|,$$
$$|\Delta v| < a\varepsilon, \quad \textit{where } v \textit{ is defined.}$$

PROOF. The mapping $z = e^{\zeta}$ transports $h(z)$ to a function $\tilde{h}(\zeta)$ which satisfies the hypotheses of Lemma 7. Since $\tilde{h}$ is periodic, the proof of Lemma 7 furnishes a function $\tilde{v}$ which is also periodic. Thus $\tilde{v}(\zeta)$ pulls back to a function $v(z)$ which solves our problem. We note that $|\Delta v| \leqslant |\Delta \tilde{v}|$. This lemma could also have been proved directly by passing to polar coordinates and solving an appropriate boundary value problem.

LEMMA 10. *Let K_1 and K_2 be compact subsets of a Riemann surface R and let U_1 and U_2 be neighborhoods of K_1 and K_2 respectively. Then there are neighborhoods*

$$K_j \subset G_j \subset U_j, \qquad j = 1, 2,$$

and a positive constant a such that if h is essentially harmonic on U_2 and bounded by ε on $U_1 \cap U_2$, then $h|G_2$ has an extension $\tilde{h}$ to $G_1 \cup G_2$ such that $\tilde{h}$ is C^2 on G_1 and satisfies

$$|\tilde{h}| < a\varepsilon, \qquad |\Delta\tilde{h}| < a\varepsilon, \qquad z \in G_1.$$

PROOF. For $j = 1, 2$, let G_j be a neighborhood of K_j in U_j which is bounded by finitely many disjoint analytic Jordan curves in U_j. We may assume that ∂G_1 and ∂G_2 are transversal to each other in the sense that they have no points of tangency and the angles of intersection must be positive. Let $\tilde{G}_2$ be a "swelled" neighborhood of G_2 obtained by moving each boundary curve of G_2 slightly away from G_2. We may assume $\partial\tilde{G}_2$ is analytic. Each component of $\tilde{G}_2 \setminus G_2$ is called a *collar* and we may assume that each collar is homeomorphic to an annulus. We may construct $\tilde{G}_2$ so close to G_2 that if C is any collar, then the components of $\bar{C} \cap \bar{G}_1$ are either all of $\bar{C}$ or else disjoint quadrilaterals with one side of the quadrilateral on $\partial\tilde{G}_2$, the opposite side on ∂G_2, and the other two on ∂G_1. The components of $C \cap \bar{G}_1$ are called *transition domains*. We may assume that $\tilde{G}_2$ is so close to G_2 that each transition domain is in $U_1 \cap U_2$.

We set $\tilde{h} = h$ on G_2 and $\tilde{h} = 0$ on $G_1 \setminus \tilde{G}_2$. Thus, we have only to define $\tilde{h}$ on the transition domains.

Let D be a transition domain. Then D is conformally equivalent to either an annulus or a rectangle. Suppose first that D is equivalent to an annulus $(1 - \delta < |z| < \rho + \delta)$. Let f be the conformal map. By Lemma 9 there is a constant a_1 independent of h and a C^2-function h_1 satisfying

$$h_1 = 0, \quad \text{for } |z| \leqslant 1,$$
$$|h_1| < a_1\varepsilon, \quad \text{for } 1 \leqslant |z| \leqslant \rho,$$
$$h_1 = h \circ f, \quad \text{for } \rho \leqslant |z|,$$
$$|\Delta h_1| < a_1\varepsilon, \quad \text{where } h_1 \text{ is defined.}$$

Now set $\tilde{h} = h_1 \circ f^{-1}$ on D. Then $\tilde{h}$ has the necessary properties and $\Delta\tilde{h} = \Delta h_1/|f'|^2$ is appropriately bounded since f' is bounded away from zero on $1 \leqslant |z| \leqslant \rho$. The bound $a(D)$ which we obtain depends only on D and not on h. Thus, $\tilde{h}$ has been defined on D in case D is homeomorphic to an annulus.

If D is not an annulus then D is a "quadrilateral" with one side α_1 on ∂G_2 and the opposite side α_0 on $\partial\tilde{G}_2$. There are positive constants x_0 and y_0 and δ and a conformal map f of the rectangle $(-\delta < x < x_0 + \delta, 0 < y < y_0)$ onto D such that the left and right boundaries correspond to α_0 and α_1 respectively. As in the annular case, but using Lemma 8 instead, there is a function h_1 satisfying

$$h_1 = 0, \quad \text{for } x \leqslant 0,$$
$$|h_1| < a_1\varepsilon, \quad \text{for } 0 \leqslant x \leqslant x_0, 0 < y < y_0,$$
$$h_1 = h \circ f, \quad \text{for } x_0 \leqslant x, 0 < y < y_0,$$
$$|\Delta h_1| < a_1\varepsilon, \quad \text{where } h_1 \text{ is defined.}$$

Again, set $\tilde{h} = h_1 \circ f^{-1}$ on D. To see that

$$\Delta\tilde{h} = \Delta h_1/|f'|^2$$

is appropriately bounded, we have only to look at f' on

$$Q = (0 \leqslant x \leqslant x_0, 0 \leqslant y \leqslant y_0),$$

for Δh_1 is zero elsewhere. Thus, it is of no consequence that f' may approach zero. We need only the fact that f' is bounded away from zero on Q for by the symmetry principle f extends conformally across the boundaries

$$(-\delta < x < x_0 + \delta, y = 0) \quad \text{and} \quad (-\delta < x < x_0 + \delta, y = y_0).$$

Hence we have extended $\tilde{h}$ to each transition domain and we have found an associated constant. Let a equal the maximum of these constants as we vary over the finitely many transition domains. This completes the proof.

LEMMA 11. *Let R be an open Riemann surface and let K_1, K_2, U_1, U_2 be as in the previous lemma. Then there is a positive constant a such that if h is essentially harmonic on U_2 and bounded by ε on $U_1 \cap U_2$, then there is a function r essentially harmonic on R satisfying*

$$|r| < a\varepsilon, \quad \text{on } K_1,$$
$$|r - h| < a\varepsilon, \quad \text{on } K_2.$$

PROOF. Let a_1 be the constant of the previous lemma. If $\tilde{h}$ is as in the previous lemma then by the Green formula, we may write

$$\tilde{h} = r(z) - \frac{1}{2\pi} \iint_{G_1 \cup G_2} g(\zeta, z)\Delta\tilde{h}(\zeta)\, d\xi\, d\eta$$

where g is the Green function for $G_1 \cup G_2$ and r is essentially harmonic on $G_1 \cup G_2$. We set

$$a = a_1 + \sup_{K_1 \cup K_2} a_1 \iint_{G_1 \cup G_2} g(\zeta, z)\, d\xi\, d\eta.$$

Then a and r have the required properties, except that r is only essentially harmonic on $G_1 \cup G_2$. However, r has only finitely many singularities and so by the Runge-type lemma, we may assume that r is essentially harmonic on all of R. This proves the lemma.

At last we may state the

FUSION LEMMA 12. *Let K_1 and K_2 be compact sets in an open Riemann surface R and let V be an open neighborhood of $K_1 \cap K_2$. There is a positive constant a such that if q_1 and q_2 are essentially harmonic functions on R satisfying for some $\varepsilon > 0$,*

$$|q_1 - q_2|_V < \varepsilon,$$

then there is a function r, essentially harmonic on R, such that for j = 1, 2,

$$|r - q_j|_{K_j} < a\varepsilon.$$

REMARK 1. Clearly, if $K_1 \cup K_2$ is full in R and if q_1 and q_2 are harmonic on $K_1 \cup K_2$, we may assume that r is actually harmonic on R.

PROOF. Let U_1 and U_2 be open neighborhoods of K_1 and K_2 respectively such that $U_1 \cap U_2 = V$. Set $q = q_1 - q_2$. Then q satisfies the hypotheses of the previous lemma, and so there exists a function s essentially harmonic on R and satisfying

$$|s| < a\varepsilon, \quad z \in K_1; \qquad |s - q| < a\varepsilon, \quad z \in K_2.$$

Set $r = q_1 - s$. Then r has the required properties and the proof is complete.

4. Necessity. Before proving our theorems, we shall discuss the interdependence of some of our conditions.

REMARK 1. Conditions (II) and (III) together imply $\partial F = \partial \hat{F}$.

PROOF. We observe that $\partial \hat{F} \subset \partial F$ and that $\partial F \setminus \partial \hat{F} = \partial F \cap (\hat{F})^0$. We show that $\partial F = \partial \hat{F}$. For suppose, to obtain a contradiction, that $p \in (\hat{F})^0 \cap \partial F$. Since $p \in (\hat{F})^0$ we may choose a closed bounded ball B with $p \in B^0$, $B \subset \hat{F}$. Let α_j, $j = 1, 2, \ldots,$ be the (at most countably many) components of $\partial B \setminus F$. For each j, let U_j be the component of $R \setminus F$ containing α_j. Since $p \in (\hat{F})^0$, we may assume that B is so small that each U_j is bounded. We now set

$$V = B^0 \cup \bigcup_{j=1}^{\infty} U_j.$$

From condition (III) it follows that V is bounded and clearly $\partial V \subset F$. Thus V violates condition (II), which is the desired contradiction. We have shown that $\partial F = \partial \hat{F}$.

REMARK 2. (III) $\rightarrow$ (I).

From the previous remarks, we have only to show the necessity of (III) and (II) for Theorem 2.

PROOF OF NECESSITY OF (II). Suppose V is a bounded open set such that $\partial V \subset F$, $x_1 \in V \cap F$ and $x_0 \in V \setminus F$. We may assume that V is connected. The sufficiency proofs to come (which generalize Brelot-Deny and which are independent of necessity proofs) show there is a harmonic f on $R \setminus \{x_0\}$ which is less than 0 on ∂V and more than 2 at x_1. Suppose f can be approximated within 1 on F by h harmonic on R. Then h violates the maximum principle in V. This proves the necessity of (II).

To complete the necessity in Theorem 2, we have only to show condition (III) is necessary. Suppose then that approximation is possible, and to obtain a contradiction, we suppose that condition (III) fails. Then for some compact $K \subset R$, $R \setminus (F \cup K)$ has bounded components which reach arbitrarily far out. To be more precise, we may exhaust R by full domains G_n in such a way that for $n = 1, 2, \ldots,$

$$\overline{G}_n \subset G_{n+1},$$

and $(R \setminus F) \setminus \bar{G}_1$ has components D_n, with

$$D_n \cap G_2 \neq \varnothing, \qquad D_n \subset G_{5n}, \qquad D_n \not\subset G_{5n-1}.$$

For $n = 3, 4, \ldots,$ choose

$$a_n \in D_n \cap \partial G_2, \qquad b_n \in D_n \cap \partial G_{5n-1}.$$

We may assume that a_n converges to some point $a \in \partial G_2$. Of course b_n tends to the ideal boundary.

Let U_n be the component of $D_n \setminus \bar{G}_{5n-4}$ which contains b_n. Also, let ω_n be the harmonic measure for D_n at a_n, considered as a measure on ∂D_n. We shall construct a sequence f_n with the following properties:

$$f_n \text{ is harmonic on } R \setminus b_n, \tag{1}$$

$$|f_n(p)| < 2^{-n}, \qquad p \in \bar{G}_{5n-5}, \tag{2}$$

$$\sum_{j=1}^{n} f_j(p) > 2\left(n + \left\| \sum_{j=1}^{n-1} f_j \right\|_{D_n}\right) \omega_n(\partial U_n \setminus G_{5n-1})^{-1}, \tag{3}$$

for $p \in E_n$, where E_n is some subset of $\partial U_n \setminus G_{5n-1}$ whose ω_n-measure exceeds

$$\omega_n(\partial U_n \setminus G_{5n-1})/2.$$

$$f_n(p) \geqslant 0, \qquad p \in \partial \bar{D}_n. \tag{4}$$

Suppose for the moment that such a sequence f_n has been constructed. Then by (2), Σf_n converges to some f harmonic on F. Suppose, to obtain a contradiction, that there is a g harmonic on R with $\|f - g\|_F < 1$. Then

$$\begin{aligned} g(a_n) &= \int_{\partial D_n} g \, d\omega_n = \int_{\partial D_n} (g - f) \, d\omega_n + \int_{\partial D_n} f \, d\omega_n \\ &> \int_{\partial D_n} f \, d\omega_n - 1 = \int_{E_n} f \, d\omega_n + \int_{\partial D_n \setminus E_n} f \, d\omega_n - 1 \\ &\geqslant \int_{E_n} \sum_1^n f_j \, d\omega_n + \int_{E_n} \sum_{n+1}^{\infty} f_j \, d\omega_n + \int_{\partial D_n \setminus E_n} \sum_1^{\infty} f_j \, d\omega_n - 1 \\ &\geqslant 2\left(n + \left\| \sum_1^{n-1} f_j \right\|_{D_n}\right) \omega_n(\partial U_n \setminus G_{5n-1})^{-1} \omega_n(\partial U_n \setminus G_{5n-1})/2 \\ &\quad - \frac{1}{2^n} - \left\| \sum_1^{n-1} f_j \right\|_{D_n} + \int_{\partial D_n \setminus E_n} f_n \, d\omega_n - \frac{1}{2^n} - 1 \geqslant n - 2. \end{aligned}$$

We have seen that $g(a_n) \geqslant n - 2$, and since a_n converges to a, this contradicts the harmonicity of g at a. Hence, the proof is complete, modulo the construction of the sequence $\{f_n\}$.

Set $f_1 = 0$, and suppose f_j have been constructed for $j = 1, 2, \ldots, n - 1$. Let A_n be the envelope of $\bar{G}_{5n-4} \cup \partial U_n$ with respect to $R \setminus \{b_n\}$. Now set

$$K_n = \bar{G}_{5n-4} \cup \partial \hat{A}_n = \bar{G}_{5n-4} \cup \alpha_n,$$

where α_n is part of $\partial U_n \setminus \bar{G}_{5n-4}$. We define $\varphi = \varphi_n$ on K_n as follows:

$$\varphi = 2\left(n + \left\| \sum_1^{n-1} f_j \right\|_{D_n}\right)\omega_n(\alpha_n \setminus G_{5n-1})^{-1} + \left\| \sum_1^{n-1} f_j \right\|_{D_n} + 2, \tag{5}$$

on $\alpha_n \setminus G_{5n-1}$.

$$\varphi = 2^{-(n+1)}, \quad \text{on } K_n \cap G_{5n-2}. \tag{6}$$

(7) φ is extended continuously to the rest of α_n (and hence to K_n) without changing its upper and lower bounds.

The function φ is continuous on K_n and harmonic on K_n^0.

We claim that ∂K_n is stable. Indeed, we have only to verify that the complement of K_n is not thin at any boundary point of K_n. This follows from [6, p. 216] and since K_n is full in $R \setminus \{b_n\}$.

Since ∂K_n is stable, we may approximate φ by a function harmonic on K_n (Lemma 1), and since K_n is full in $R \setminus \{b_n\}$, we may even, according to the Pfluger-Runge theorem (Lemma 6), approximate by a function f_n harmonic on $R \setminus \{b_n\}$. We may approximate so well that f_n satisfies (1), (2), (3), and (4), and the proof of necessity is complete for Theorem 2.

To prove the necessity in Theorem 4, suppose $R^* \setminus G$ is not connected. Then there is a compact curve γ bounding an open set V which contains a point of G and a point of $R \setminus G$. Then the proof of necessity of (II) now applies.

5. Sufficiency. We shall prove simultaneously Theorem 1 and the sufficiency in Theorem 3.

For any of our theorems we introduce the *special case* of that theorem. By the special case of Theorem n we mean Theorem n restricted to pairs (F, R) such that R has an essential extension in which F is bounded. We shall now show that in order to prove Theorem 1 and the sufficiency in Theorem 3, it is sufficient to do so for the special cases.

Suppose, then, that the special cases of Theorem 1 and the sufficiency in Theorem 3 have already been established and let h be essentially harmonic on F. Let $\{M_j\}$ be a locally finite cover of F by disjoint open sets of finite genus. We may assume h is defined on each M_j. Let $\{R_n\}$ be an exhaustion of R with the property that

$$R_j \cap M_k = \varnothing, \qquad k \geqslant j.$$

If $R^* \setminus F$ is connected and locally connected, we may further assume that the exhaustion is compatible with F.

Let h be essentially harmonic on F. Since $R_1 \cup M_1$ has finite genus, it has a compact extension by a theorem of Bochner [1]. Thus, if $\varepsilon > 0$, then by the special case of Theorem 1, there is a u_1 essentially harmonic on $R_1 \cup M_1$ such that

$$|u_1 - h|_{F \cap M_1} < \varepsilon/2.$$

Moreover, if $R^* \setminus F$ is connected and locally connected and if h is harmonic on F, then by the special case of Theorem 3, we may take u_1 to be harmonic on $R_1 \cup M_1$.

Set $R_0 = \varnothing$. Again by the special case of Theorem 1, there is a u_2 essentially harmonic on

$$R_2 \cup \bigcup_{j=1}^{2} M_j$$

such that

$$|u_2 - u_1| < \varepsilon/2^2, \quad \text{on } \bar{R}_0 \cup (F \cap M_1),$$

and

$$|u_2 - h| < \varepsilon/2^2, \quad \text{on } F \cap M_2.$$

We proceed inductively to construct u_n, $n = 2, 3, \ldots,$ essentially harmonic on

$$R_n \cup \bigcup_{j=1}^{n} M_j,$$

such that

$$|u_n - u_{n-1}| < \varepsilon/2^n, \quad \text{on } \bar{R}_{n-2} \cup \left\{ F \cap \bigcup_{j=1}^{n-1} M_j \right\},$$

and

$$|u_n - h| < \varepsilon/2^n, \quad \text{on } F \cap M_n.$$

Note that u_n converges to a function u essentially harmonic on R, and that

$$|u - h| < \varepsilon/2^{n-1}, \quad \text{on } F \cap M_n.$$

Thus, our approximation is actually somewhat better than uniform. In the context of Theorem 3, we may assume that each u_n is harmonic so that u is also. This completes the proof of Theorem 1 and the sufficiency in Theorem 3 modulo the special cases.

Before proving the special cases, we note that we may assume h is actually harmonic on F, for by the Cousin-type theorem, there is a function h_1, essentially harmonic on R, such that $h_1 - h$ is harmonic on F. Suppose we can find u_1 essentially harmonic on R such that u_1 approximates $h - h_1$ on F. Then $u_1 + h_1$ approximates h.

To prove the special cases, suppose R has an essential extension R' in which F is bounded. Let h be harmonic on F and fix $\varepsilon > 0$. We may assume that h is harmonic on a (possibly infinite) polygonal neighborhood U of F. We may assume that if $R^* \setminus F$ is connected and locally connected, the same holds for $R^* \setminus \bar{U}$ (see [4, p. 152]). Let $\{R_n\}$ be an exhaustion of R. If $R^* \setminus \bar{U}$ is connected and locally connected, we may assume that $\{R_n\}$ is compatible with $\bar{U}$. For each n, set $U_n = U \cap R_n$. Let $\bar{F}$ be the R'-closure of F and let $\tilde{R}$ be an open Riemann surface with $R \cup \bar{F} \subset \tilde{R} \subset R'$. We apply the Fusion Lemma to the Riemann surface $\tilde{R}$, replacing K_1, K_2, V respectively by $\bar{R}_n$, $\bar{F} \setminus R_n$, U_{n+1}. We may choose the $\{a_n\}$ so that $1 < a_n < a_{n+1}$. We select the positive numbers $\varepsilon_1, \varepsilon_2, \ldots,$ so that

$$\varepsilon_{n+1} < \varepsilon_n \quad \text{and} \quad \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon/2.$$

By Lemma 6, there exist essentially harmonic functions q_n on $\tilde{R}$ such that

$$|q_n - h| < \varepsilon_n / 2a_n, \quad \text{on } \bar{U}_{n+1}, \tag{1}$$

and therefore,

$$|q_{n+1} - q_n| < \varepsilon_n / a_n, \quad \text{on } \bar{U}_{n+1}, n = 1, 2, \ldots .$$

Moreover, if $R^* \setminus \bar{U}$ is connected and locally connected, the same is true of $\tilde{R}^* \setminus \bar{U}$ and we may choose each q_n to be harmonic. By the Fusion Lemma, for each $n = 1, 2, \ldots,$ there exists an essentially harmonic function r_n on $\tilde{R}$ such that

$$|r_n - q_n| < \varepsilon_n, \quad \text{on } \bar{R}_n, \tag{2}$$

$$|r_n - q_{n+1}| < \varepsilon_n, \quad \text{on } \bar{F} \setminus R_n. \tag{3}$$

Moreover, if $R^* \setminus \bar{U}$ is connected and locally connected, we may assume that each r_n is harmonic. The inequalities (2) yield

$$\sum_n^\infty |r_\nu - q_\nu| < \sum_n^\infty \varepsilon_\nu, \quad \text{on } \bar{R}_n.$$

Therefore,

$$u = q_1 + \sum_1^\infty (r_\nu - q_\nu)$$

is essentially harmonic on $R = \bigcup_{n=1}^\infty R_n$.

Set $F_n = F \cap \bar{R}_n$. From (1) and (2), there follows on F_1,

$$|u - h| \leq |q_1 - h| + \sum_1^\infty |r_\nu - q_\nu| < \frac{\varepsilon_1}{2a_1} + \sum_1^\infty \varepsilon_\nu < \varepsilon.$$

From (3), (1), and (2), we also have

$$|u - h| \leq \sum_1^{n-1} |r_\nu - q_{\nu+1}| + |q_n - h| + \sum_n^\infty |r_\nu - q_\nu|$$
$$< \sum_1^{n-1} \varepsilon_\nu + \frac{\varepsilon_n}{2a_n} + \sum_n^\infty \varepsilon_\nu < \varepsilon,$$

on $F_n \setminus F_{n-1}, n = 2, 3, \ldots .$

Thus, u can be approximated uniformly on F by functions essentially harmonic on R, and by harmonic functions in case $R^* \setminus F$ is connected and locally connected. This completes the proof of Theorem 1 and the proof of sufficiency in Theorem 3.

Let us now prove the sufficiency in Theorem 4. Actually this is a corollary of the sufficiency in Theorem 3. Indeed, suppose G is open in the open Riemann surface R, that G is essentially of finite genus, and that $R^* \setminus G$ is connected. Let h be harmonic on G; let F be a closed subset of R contained in G; and let $\varepsilon > 0$. We may construct a locally finite, closed, polygonal neighborhood P of F in G with $R^* \setminus P$ connected. Since ∂P is locally finite, it is clear that $R^* \setminus P$ is also locally connected. From Theorem 3, there is a function u harmonic on R such that $|u - h| < \varepsilon$ on P. This completes the proof of sufficiency in Theorem 4.

We shall now prove the sufficiency in Theorem 5. Let h be continuous on F and harmonic on F°. Let $\{p_n\}$ be chosen, one from each component of $R - F$. Since ∂F is analytic, $\{p_n\}$ is closed. Set $\tilde{R} = R \setminus \bigcup_{n=1}^{\infty}\{p_n\}$. Then ∂F satisfies the hypotheses of Saginjan's theorem **[11]** in $\tilde{R}$. We note that Saginjan's theorem holds also on open Riemann surfaces. Thus, there is a harmonic function v on $\tilde{R}$ such that

$$|v - h| < \varepsilon/2 \quad \text{on } \partial F,$$

where ε is a prescribed positive number. Let H_{v-h} denote the Perron-Wiener-Brelot solution of the Dirichlet problem for F° with boundary values $u - h$. Then, on F, $-\varepsilon/2 \leqslant H_{v-h} \leqslant \varepsilon/2$. Let $u = v - h - H_{v-h}$. Then $u = 0$ on ∂F. Assuming that ∂F is analytic, u extends to a harmonic function on F which we shall continue to denote by u. Hence, we are back in the Runge case and therefore there exists an essentially harmonic function r on R such that

$$|u - r| < \varepsilon/4 \quad \text{on } F.$$

If $R^* \setminus F$ is connected, we may take r to be harmonic on R. Thus, on F we have

$$\begin{aligned}|(v - r) - h| &= |\{(v - u) - h\} + (u - r)| \\ &\leqslant |(v - u) - h| + |u - r| = |H_{v-h}| + |u - r| \\ &< \varepsilon/2 + \varepsilon/4 = 3\varepsilon/4.\end{aligned}$$

If $R^* \setminus F$ is connected, $v - r$ is the required harmonic function on R which approximates h. If $R^* \setminus F$ is not connected, we may, by Theorem 1, approximate v within $\varepsilon/4$ on F by a function v_1 essentially harmonic on R. In this case, $v_1 - r$ yields the required approximation.

This paper has been concerned mostly with Runge-type approximation. We included Theorem 5 on Walsh-type approximation because it is easily derived from the Runge-type theorem. The assumptions in Theorem 5 can be somewhat relaxed, but this will be discussed in a subsequent paper dealing directly with Walsh-type approximation.

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